The orthogonality relations for the supergroup $U(m|n)$

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Abstract

Starting from the generalization of the Itzykson-Zuber integral for $U(m|n)$ we determine the orthogonality relations for this supergroup.

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Motivated by the recent progress made in the study of random surfaces and statistical systems on random surfaces,which might have important applications in non-critical string theory as well as Quantum Chromodynamics (QCD) in the large N limit, we have considered the extension of some of these ideas to the case where the associated random matrices [1] are replaced by supermatrices. An important mathematical object that appears naturally in the discussion of random matrices is the Itzykson-Zuber (IZ) integral over the unitary group [2]. This integral has been applied to the solution of the two-matrix model [2], [3] and, more recently, to the Migdal-Kazakov model of "induced QCD" [4].

Recently we have extended the IZ integral to the case of the unitary supergroup $U(m|n)$ [5]. In this letter we apply this result to determine the orthogonality relations among irreducible representations of this supergroup. The basic problem that arises is that the integration measure $[dU]$ over $U(m|n)$ is of the Berezin type, which include integrations over odd Grassmann numbers according to the standard recipe [6]. Thus, in many cases the integration over $[dU]$ of supermatrix elements corresponding to arbitrary representations of the supergroup will be automatically zero due to the above mentioned Grassmannian character. In particular, this will happen in the case of the orthogonality relations and the purpose of this letter is to characterize the irreducible representations of $U(m|n)$ which lead to a non-zero result together with the determination of the corresponding normalization coefficient.

In the following paragraphs we briefly summarize our conventions regarding representations of supergroups together with some results that will be used subsequently.

Supergroups will be represented by linear operators $\tilde{D}(g)$ acting on some vector space with basis $\{\Phi_I\}$. Linearity is defined by $\tilde{D}(g)\left(\Phi_I\alpha + \Phi_J\beta\right) =$

$\left(\tilde{D}(g)\Phi_I\right)\alpha + \left(\tilde{D}(g)\Phi_J\right)\beta$, where α and β are arbitrary Grassmann numbers. The action

$$\tilde{D}(g)\left(\Phi_I\right) = \sum_J^{m_t+n_t} \Phi_J \mathcal{D}^{(t)}_{JI}(g), \tag{1}$$

defines a representation $\{t\}$ of the supergroup characterized by the Young tableau $(t_1, t_2, \ldots)$, with $t_1 \geq t_2 \geq, \ldots$, in the usual notation. Here $\mathcal{D}^{(t)}_{JI}(g)$ are the elements of an $(m_t + n_t) \times (m_t + n_t)$ supermatrix written in the standard block form [6]. In fact, our definition of linearity given above guarantees that the definition (1) satisfies $\mathcal{D}^{(t)}_{JI}(g_1 * g_2) = \sum_K \mathcal{D}^{(t)}_{JK}(g_1)\mathcal{D}^{(t)}_{KI}(g_2)$, thus providing a representation of the supergroup in terms of the standard multiplication of supermatrices.

The Schur's lemmas can be directly proved in the case of finite supergroups and the extension to continuous supergroups is made in complete analogy to the classical case. In general, the corresponding measure must be left and right-invariant under the supergroup action and for the case of $U(m|n)$ it is defined by $[dU] = \mu \prod_{P,Q=1}^{m+n} dU_{PQ} dU^*_{PQ}\ \delta(UU^\dagger - I)$, where the δ-function really means the product of $(m+n)^2$ unidimensional δ-functions corresponding to the independent constraints set by the condition $UU^\dagger = I$. The normalization constant μ is fixed by our normalization of the supersymmetric IZ integral. It is important to observe that the above measure possesses $2mn$ real independent odd differentials.

The application of Schur's lemmas to the quantity $\mathcal{X}^{st}_{IJ} = \int [dU] \mathcal{D}^{(s)}_{IL}(U)$ $X_{LM} \mathcal{D}^{(t)}_{MJ}(U^{-1})$, where X_{LM} is an arbitrary supermatrix, leads to the conclusion that $\mathcal{X}^{st}_{IJ}$ must be a multiple of the identity supermatrix. Factoring out the arbitrary piece X_{LM}, we are left with the ortogonality relations

$$\int [dU] \mathcal{D}^{(s)}_{IJ}(U) \mathcal{D}^{(t)*}_{KL}(U) = (-1)^{\epsilon_J} \alpha_{\{t\}} \delta^{st} \delta_{IK} \delta_{JL}, \tag{2}$$

where $(U^\dagger)_{ij} = (U^{-1})_{ij} = (U^*)_{ji}$. Let us recall that there are two fundamental representations of $U(m|n)$: $\mathcal{D}_{ij}(U) = U_{ij}$ and $\dot{\mathcal{D}}_{ij}(U) = \tilde{U}_{ij} = (-1)^{\epsilon_j+\epsilon_j\epsilon_i}(U^*)_{ij}$. These lead to three types of irreducible representations $\{t\}$: undotted $\{u\}$, dotted $\{\dot{v}\}$ and mixed $\{\dot{v}\}|\{u\}$ [7].

As compared to the classical case, the appearence of odd integration variables in $[dU]$ imposes further contraints upon the representations that give a non-zero value for the coefficient $\alpha_{\{t\}}$ in (2). The main result of this letter is to characterize such representations in the dotted and undotted cases.

To begin with, we prove that they satisfy the following Lemma: The supercharacters $s\chi_{\{t\}}(U) \equiv \sum_I (-1)^{\epsilon_I}\mathcal{D}^{(t)}_{II}(U)$ of the representations $\mathcal{D}^{(t)}_{IJ}(U)$ for which $\alpha_{\{t\}} \neq 0$ constitute a linearly independent set. The proof goes as follows: the ortogonality relations (2) imply that

$$\int [dU] s\chi_{\{s\}}(U)\mathcal{D}^{(t)*}_{KL}(U) = \alpha_t \delta^{st}\delta_{KL}. \tag{3}$$

Next, let us consider a null linear combination of supercharacters of representations with $\alpha_{\{s\}} \neq 0$: $\sum_s a_s s\chi_{\{s\}}(U) = 0$. Multiplying this equation by $\mathcal{D}^{(t)*}_{kl}(U)$ and integrating over dU we have $a_t\alpha_{\{t\}}\delta_{kl} = 0$ for each representation $\{t\}$, which shows that $a_t = 0$ provided $\alpha_{\{t\}} \neq 0$.

The starting point that leads to the determination of the undotted representations $\{t\}$ together with the values of the non-zero $\alpha_{\{t\}}$ in (2) is our supersymmetric extension of the IZ integral given by [5]

$$\begin{aligned}\tilde{I}(M_1, M_2; \beta) &\equiv \int [dU] e^{\beta Str(M_1 U M_2 U^\dagger)} \\ &= \beta^{mn}\Sigma(\lambda_1, \bar{\lambda}_1)\Sigma(\lambda_2, \bar{\lambda}_2) I(\lambda_1, \lambda_2; \beta) I(\bar{\lambda}_1, \bar{\lambda}_2; -\beta),\end{aligned} \tag{4}$$

where $I(d_1, d_2; \gamma)$ is the standard IZ integral [2]

$$I(d_1, d_2; \gamma) = \gamma^{-\frac{N(N-1)}{2}} \prod_{p=1}^{m-1} p! \frac{det(e^{\gamma d_{1i} d_{2j}})}{\Delta(d_1)\Delta(d_2)}, \tag{5}$$

and

$$\Delta(d) = \prod_{i>j}(d_i - d_j), \qquad \Sigma(\lambda, \bar{\lambda}) = \prod_{i=1}^{m}\prod_{\alpha=1}^{n}(\lambda_i - \bar{\lambda}_\alpha). \tag{6}$$

Here M_1 and M_2 are $(m+n)\times(m+n)$ hermitian supermatrices which can be diagonalized [8] and β is a complex parameter. Our notation is such that the first m eigenvalues of M are identified by λ_i, while the remaining n eigenvalues are denoted by $\bar{\lambda}_\alpha$. Such partition is characterized by the following parity assignment of the eigenvector components $V_P, \bar{V}_P : \epsilon(V_P) = \epsilon(P), \epsilon(\bar{V}_P) = \epsilon(P) + 1$.

A convenient way of rewriting the standard IZ integral is in terms of its expansion in characters of the corresponding irreducible representations of the group $U(m)$, [2]

$$I(\lambda_1, \lambda_2; \beta) = \sum_{\{n\}} \frac{\beta^{|n|}}{|n|!}\frac{\sigma_{\{n\}}}{d_{\{n\}}}\chi_{\{n\}}(\lambda_1)\chi_{\{n\}}(\lambda_2). \tag{7}$$

Following analogous steps, we obtain the supercharacter expansion of the expresion (4)

$$\tilde{I}(M_1, M_2; \beta) = \sum_{\{t\}} \frac{\beta^{|t|}}{|t|!}\sigma_{\{t\}}\alpha_{\{t\}} s\chi_{\{t\}}(M_1) s\chi_{\{t\}}(M_2), \tag{8}$$

which contains only undotted representations. The above relation has been obtained without the use of a completeness relation for the supercharacters. Here $|t|$ denotes the total number of boxes in the Young tableau corresponding to the irreducible representation $\{t\}$ of $U(m|n)$ and $\sigma_{\{t\}}$ counts the number of times that this representation is contained in the tensor product $\otimes^{|t|}\mathcal{D}$.

In virtue of the Lemma previously proved, we see that the representations which contribute to Eq.(8) have supercharacters that form a linearly independent set.

Now we consider the determination of the representations with non-zero $\alpha_{\{t\}}$. The basic expression we use is the character expansion in both sides of Eq.(4), which is

$$\sum_{\{t\}} \frac{\beta^{|t|}}{|t|!} \sigma_{\{t\}} \alpha_{\{t\}} s\chi_{\{t\}}(M_1) s\chi_{\{t\}}(M_2) = \sum_{\{p\}} \sum_{\{q\}} \frac{\beta^{|p|+|q|+mn}}{|p|!|q|!} \frac{\sigma_{\{p\}}\sigma_{\{q\}}}{d_{\{p\}}d_{\{q\}}} (-1)^{|q|} \times$$
$$\times \Sigma(\lambda_1, \bar{\lambda}_1) \chi_{\{p\}}(\lambda_1) \chi_{\{q\}}(\bar{\lambda}_1) \Sigma(\lambda_2, \bar{\lambda}_2) \chi_{\{p\}}(\lambda_2) \chi_{\{q\}}(\bar{\lambda}_2). \quad (9)$$

Now we analize this equation by considering the following cases:

a) Case of $|t| < mn$

Before making any further analysis, from (9) we can immediately conclude that

$$\alpha_{\{t\}} = 0, \text{ for } |t| = 0, 1, \ldots, (mn-1). \quad (10)$$

because in both sides of this equation we have a power series in β, and the right term of it starts with β^{mn} while the left one starts with β^0. The proof goes by assuming that some coefficients $\alpha_{\{t\}}$ are non-zero. The linear independence of them, together with the above observation, imply that they must be zero, thus leading to a contradiction.

b) Case of $|t| \geq mn$

As we said just before, Eq.(9) is a power series in β in both sides of the equation, so for a same power of β we must have the same coefficient

$$\frac{1}{|t|!} \sum_{\{t\}} \sigma_{\{t\}} \alpha_{\{t\}} s\chi_{\{t\}}(M_1) s\chi_{\{t\}}(M_2) = \sum_{\{p\}} \sum_{\{q\}} \frac{(-1)^{|q|}}{|p|!|q|!} \frac{\sigma_{\{p\}}\sigma_{\{q\}}}{d_{\{p\}}d_{\{q\}}} \times$$
$$\times \Sigma(\lambda_1, \bar{\lambda}_1) \chi_{\{p\}}(\lambda_1) \chi_{\{q\}}(\bar{\lambda}_1) \; \Sigma(\lambda_2, \bar{\lambda}_2) \chi_{\{p\}}(\lambda_2) \chi_{\{q\}}(\bar{\lambda}_2), \quad (11)$$

where the sum in the LHS is made for all tableaux having a fixed number of boxes $|t|$, while the sum over $\{p\}$ and $\{q\}$ in the RHS is restricted to

$$|p| + |q| = |t| - mn. \tag{12}$$

We now want to prove that Eq.(11) necessarily implies that

$$s\chi_{\{t\}}(M) = c_{\{p,q\}}\Sigma(\lambda, \bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{\{q\}}(\bar{\lambda})$$

for some $\{p\}$ and $\{q\}$ satisfying (12) and for a certain representation $\{t\}$ that we will determine.

In order to extract more information from Eq.(11) let us consider an arbitrary supermatrix M_2, while we restrict the supermatrix M_1 in such a way that one of its λ-eigenvalues be equal to one of its $\bar{\lambda}$- eigenvalues. Namely, let $\lambda_j = \bar{\lambda}_\beta$, for example. Then, in Eq.(11) we are left with

$$\frac{1}{|t|!}\sum_{\{t\}} \sigma_{\{t\}}\alpha_{\{t\}}s\chi_{\{t\}}(M_1)s\chi_{\{t\}}(M_2) = 0, \tag{13}$$

because $\Sigma(\lambda_1, \bar{\lambda}_1)$ becomes zero. If we look at this relation as a null linear combination of the supercharacters $s\chi_{\{t\}}(M_2)$ with coefficients

$$\gamma_{\{t\}} = \frac{1}{|t|!}\sigma_{\{t\}}\alpha_{\{t\}}s\chi_{\{t\}}(M_1), \tag{14}$$

we conclude that the coefficients $\gamma_{\{t\}}$ are all zero, because the supercharacters appearing in (13) constitute a linearly independent set. But $\sigma_{\{t\}}$ and $\alpha_{\{t\}}$ are different from zero, so that we are left with $s\chi_{\{t\}}(\tilde{M}_1) = 0$. Recalling that $s\chi_{\{t\}}(M)$ is a polynomial function of the eigenvalues λ_i, $\bar{\lambda}_\alpha$, we conclude from this relation that $s\chi_{\{t\}}(M)$ must be divisible by $(\lambda_j - \bar{\lambda}_\beta)$. That is to say

$$s\chi_{\{t\}}(M) = (\lambda_j - \bar{\lambda}_\beta)F_{j\beta}(\lambda, \bar{\lambda}), \tag{15}$$

where $F_{j\beta}(\lambda, \bar{\lambda})$ is another polynomial function of the eigenvalues. The same reasoning can be extended to every λ_i (i=1,...,m) and $\bar{\lambda}_\alpha$ ($\alpha = 1$,...,n), and this implies that $s\chi_{\{t\}}(M)$ must have the form

$$s\chi_{\{t\}}(M) = \prod_{i=1}^{m}\prod_{\alpha=1}^{n}(\lambda_i - \bar{\lambda}_\alpha)\ P(\lambda, \bar{\lambda}) = \Sigma(\lambda, \bar{\lambda})\ P(\lambda, \bar{\lambda}). \tag{16}$$

In Eq.(16), $P(\lambda, \bar{\lambda})$ must be a homogeneous polynomial function of all the eigenvalues, because $s\chi_{\{t\}}(M)$ and $\Sigma(\lambda, \bar{\lambda})$ are so. The degree of homogeneity of $s\chi_{\{t\}}(M)$ and $\Sigma(\lambda, \bar{\lambda})$ is $|t|$ and mn, respectively. This means that the degree of homogeneity of $P(\lambda, \bar{\lambda})$ must be $|t| - mn$. Also, we know that $s\chi_{\{t\}}(M)$ and $\Sigma(\lambda, \bar{\lambda})$ are symmetric functions in the eigenvalues λ_i, $\bar{\lambda}_\alpha$, separately, and so should be $P(\lambda, \bar{\lambda})$. Summing up then, $P(\lambda, \bar{\lambda})$ is: (i) an homogeneous polynomial function of degree $|t| - mn$ in all the eigenvalues and (ii) a symmetric function of $\{\lambda_i\}$ and $\{\bar{\lambda}_\alpha\}$, separately. Since the characters $\chi_{\{a\}}(\lambda)$ $\left(\chi_{\{b\}}(\bar{\lambda})\right)$ are polynomial homogeneous functions of degree $|a|$ $(|b|)$, which are symmetric in the eigenvalues λ_i (λ_α) and constitute a complete linearly independent set, $P(\lambda, \bar{\lambda})$ can be written as

$$P(\lambda, \bar{\lambda}) = \sum_{\{a\},\{b\}} c^{\{t\}}_{\{a,b\}}\chi_{\{a\}}(\lambda)\chi_{\{b\}}(\bar{\lambda}), \tag{17}$$

where the sum in $\{a\}$ and $\{b\}$ is rectricted by $|a|+|b| = |t|-mn$. Substituing this last relation in (16) we have

$$s\chi_{\{t\}}(M) = \Sigma(\lambda, \bar{\lambda})\sum_{\{a\},\{b\}} c^{\{t\}}_{\{a,b\}}\chi_{\{a\}}(\lambda)\chi_{\{b\}}(\bar{\lambda}). \tag{18}$$

Using the above expression in the LHS of (11) and comparing both sides of this equation, we obtain that the expansion in (18) must include only one coefficient, for a given tableaux $\{t\}$, which precise form is yet to be determined. That is

$$s\chi_{\{t\}}(M) = c^{\{t\}}_{\{p,q\}}\Sigma(\lambda, \bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{\{q\}}(\bar{\lambda}), \tag{19}$$

where $\{p\}$ and $\{q\}$ satisfy (12). As the number of solutions to Eq.(12) is $|t|-mn+1$, the supercharacter expansion of the supersymmetric IZ integral will contain only ($|t|-mn+1$) terms, for a given $|t|$.

b.1) The case of $\{p\}=\{q\}=0$

Here we have $|t|=mn$ and

$$s\chi_{\{t\}}(M)=c^{\{t\}}_{\{0,0\}}\Sigma(\lambda,\bar{\lambda}). \tag{20}$$

In order to proceed with the required identifications, let us consider the particular case where the only non-zero block of the supermatrix M is the $m\times m$ block, i.e.

$$M=\left(\begin{array}{cc} M' & 0 \\ 0 & 0 \end{array}\right). \tag{21}$$

Then Eq. (20) reduces to

$$\chi_{\{t\}}(M')=c^{\{t\}}_{\{0,0\}}(\prod_{i=1}^{m}\lambda_i)^n. \tag{22}$$

Using Weyl's formula for the character of the representations of the unitary group

$$\chi_{\{r\}}(\lambda)=\frac{det({\lambda_i}^{r_j+n-j})}{det({\lambda_i}^{n-j})} \tag{23}$$

we conclude that the product of eigenvalues in (22) corresponds to the character of the representation $\{r\}=(r_1,r_2,\ldots,r_m)$ with $r_i=n$ of $U(m)$. We are using the standard notation $(r_1,r_2,\ldots,r_m)$ to denote a Young tableau with m rows, such that the i-th row has r_i boxes. In this way we have that $\chi_{\{t\}}(M')=c_{\{0,0\}}\chi_{(n,n,\ldots,n)}(M')$, which allows the identification of the representation $\{t\}$ as the one given by the tableau corresponding to $t_1=t_2=$

$\ldots = t_m = n$, together with $c^{\{t\}}_{\{0,0\}} = 1$. Besides, we identify $\Sigma(\lambda, \bar{\lambda})$ as the supercharacter of the representation referred to above. We will denote by $\{mn\}$ the representation just found, whose tableau consists of m rows, each with n boxes.

b.2) The case $\{p\} \neq 0$, $\{q\} = 0$

Here we have $|t| = |p| + mn$ and $s\chi_{\{t\}}(M) = c^{\{t\}}_{\{p,0\}} \Sigma(\lambda, \bar{\lambda}) \chi_{\{p\}}(\lambda)$. Considering in this expression the same choice of M as in (21), we have $\chi_{\{t\}}(M') = c^{\{t\}}_{\{p,0\}} (\prod_{i=1}^{m} \lambda_i)^n \chi_{\{p\}}(\lambda)$. Using again Weyl's formula we are able make the identification $(\prod_{i=1}^{m} \lambda_i)^n \chi_{\{p\}}(\lambda) = \chi_{\{n+p\}}(\lambda)$, where by $\{n + p\}$ we mean the representation with Young tableau $(n + p_1, n + p_2, \ldots, n + p_m)$. This leads to $\chi_{\{t\}}(M') = c_{\{p,0\}} \chi_{(n+p_1,n+p_2,\ldots,n+p_m)}(\lambda)$ for this case and we conclude that $c^{\{t\}}_{\{p,0\}} = 1$ with $\{t\}$ being the representation $(n + p_1, n + p_2, \ldots, n + p_m)$ of $U(m|n)$. We introduce the pictorial notation $\{n + p\} = \{mn\}\{p\}$, that will be useful in the sequel.

b.3) The case of arbitrary $\{p\}$ and $\{q\}$

Now we discuss the main result of this letter which states that the representations of $U(m|n)$ with $\alpha_{\{t\}} \neq 0$ are characterized by the following Young tableaux

$$\{\tilde{t}\} = \begin{matrix} \{mn\}\{p\} \\ \{q\}^T \end{matrix} \equiv \binom{\{p\}}{\{q\}^T}, \tag{24}$$

with the normalization coefficient given by

$$\alpha_{\{\tilde{t}\}} = (-1)^{|q|} \frac{|\tilde{t}|!}{|p|!|q|!} \frac{\sigma_{\{p\}}\sigma_{\{q\}}}{\sigma_{\{\tilde{t}\}}} \frac{1}{d_{\{p\}}d_{\{q\}}}. \tag{25}$$

The Young tableau $\{\tilde{u}\} = \binom{\{r\}}{\{s\}^T}$ introduced in (24) is constructed by starting from the basic array $\{n+r\} = \{mn\}\{r\}$ defined previously, together with the

array $\{s\} = \{s_1, s_2, \dots, s_n\}$, which is subsequently transposed and attached to the left bottom of it.

An important result that leads to the above conclusions is that

$$s\chi_{\{\tilde{t}\}}(M) = (-1)^{|q|}\Sigma(\lambda, \bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{\{q\}}(\bar{\lambda}). \tag{26}$$

Now we give some details of the proof of Eq.(26). We start from the relation

$$s\chi_{\{mn\}\{u\}}(M) = \Sigma(\lambda, \bar{\lambda})\chi_{\{u\}}(\lambda), \tag{27}$$

which is valid for every representationx $\{u\}$ of $U(m)$. The proof will follow in two steps. (i) First we prove, by induction, that

$$s\chi_{\{\tilde{t}_1\}}(M) = (-1)^{|r_1|}\Sigma(\lambda, \bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{\{r_1\}^T}(\bar{\lambda}), \tag{28}$$

where $\{\tilde{t}_1\} = \binom{\{p\}}{\{r_1\}}$ is a Young tableau of the type (24) with $\{r_1\} = (r_1)$, $r_1 \leq n$, corresponding to a single row with r_1 boxes, which is attached without transposition to the bottom of $\{mn\}\{p\}$. Let us consider first the case $(1) = \{\Box\}$, that is $r_1 = 1$. Taking $\{u\} = \{p\}$ in (27) and multiplying both sides by $s\chi_{\{\Box\}}(M) = \chi_{\{\Box\}}(\lambda) - \chi_{\{\Box\}}(\bar{\lambda})$ we obtain

$$s\chi_{\{mn\}(\{p\}\times\Box)}(M) + s\chi_{\{\tilde{t}_{11}\}}(M) = \Sigma(\lambda, \bar{\lambda})\chi_{\{p\}\times\Box}(\lambda) - \Sigma(\lambda, \bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{\{\Box\}}(\bar{\lambda}), \tag{29}$$

where $\{\tilde{t}_{11}\} = \binom{\{p\}}{(1)}$. Applying (27) to the case $\{u\} = \{p\} \times \Box$ in (29) we are left with

$$s\chi_{\{\tilde{t}_{11}\}}(M) = -\Sigma(\lambda, \bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{\{\Box\}^T}(\bar{\lambda}), \tag{30}$$

which verifies (28) for this case. Here we have made use of the Young tableux rules for multiplying representations. Next we asume that (28) is valid for the

tableau $\{\tilde{t}_{1r}\} = \binom{\{p\}}{(r)}$, and prove that it is also valid for $\{\tilde{t}_{1(r+1)}\} = \binom{\{p\}}{(r+1)}$ with $r+1 \leq n$. For this purpose we will make use of the relation [7]

$$s\chi_{(n)}(M) = \sum_{k=0}^{n} (-1)^k \chi_{(n-k)}(\lambda)\chi_{(k)^T}(\bar{\lambda}), \tag{31}$$

where we recall that (k) denotes the Young tableau having one row with k boxes while $(k)^T$ denotes de Young tableau corresponding to one column with k boxes. Considering this relation for $n = r+1$, separating the $k = r+1$ term in the summation and multiplying both sides of (27) by (31)we obtain

$$\sum_{k=0}^{r} s\chi_{\binom{\{p\}\times(r+1-k)}{(k)}}(M) + s\chi_{\binom{\{p\}}{(r+1)}}(M) = \sum_{k=0}^{r}(-1)^k \Sigma(\lambda,\bar{\lambda})\chi_{\{p\}\times(r+1-k)}(\lambda)$$
$$\times\chi_{(k)^T}(\bar{\lambda}) + (-1)^{r+1}\Sigma(\lambda,\bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{(r+1)^T}(\bar{\lambda}) \tag{32}$$

The first terms of both sides are equal (in virtue of the hipothesis of induction), so this last equation becomes the desired result. (ii) Following analogous steps we can prove by induction in r_2, that

$$s\chi_{\binom{\{p\}}{(r_1,r_2)}}(M) = (-1)^{r_1+r_2}\Sigma(\lambda,\bar{\lambda})\chi_{\{p\}}(\lambda)\chi_{(r_1,r_2)^T}(\bar{\lambda}) \tag{33}$$

for $r_2 \leq n$. The final choice $(r_1, r_2, \ldots, r_n)^T = \{q\}$ implies the proof of the relation (26), which after substitution in (11) leads to our final result (25).

An immediate consequence of our relation in (26) is that we can obtain the dimension for the representations in $U(m|n)$ that arise in the supercharacter expansion ($sd_{\{t\}}$) in terms of the dimension of representations in $U(m)$ ($d_{\{p\}}$) and $U(n)$ ($d_{\{q\}}$). Taking

$$M = \begin{pmatrix} I_{m\times m} & 0 \\ 0 & -I_{n\times n} \end{pmatrix}$$

in (26) and observing that $s\chi_{\{t\}}(M)$ becomes $sd_{\{t\}}$, we obtain the closed expression

$$sd_{\{\tilde{t}\}} = 2^{mn} d_{\{p\}} d_{\{q\}}, \tag{34}$$

for the dimensions of the representations of $U(m|n)$ characterized by the tableaux in (24). Let us also remark that our expression (25) correctly reproduces the result $\alpha_{\{t\}} = \frac{1}{d_{\{t\}}}$ for $U(n)$. We end up with a brief comment regarding the dotted and mixed representations. We are able to prove that $\alpha_{\{\dot{u}\}} = \alpha_{\{u\}}$ and also that $\alpha_{\{\dot{u}\}|\{v\}}$ can be written as a linear combination of the undotted coefficients $\alpha_{\{s\}}$. A detailed discussion of these matters will be given in a forthcoming publication.

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